Int. J. Heat Mass Transfer. Vol. 11, pp. 1413–1416. Pergamon Press 1968. Printed in Great Britain

RADIATIVE TRANSPORT IN AN OPTICALLY THICK PLANAR MEDIUM*

GEORGE EMANUEL

Aerospace Corporation, El Segundo, California, U.S.A.

(*Received* 1 *January* 1968 *and in revised form* 6 *May* 1968)

NOMENCLATURE

A,	integration constant;
E_n,	exponential integral of order n;
L,	distance between walls;
$q(\tau)$,	Hopf's function;
Q,	dimensionless heat flux, (heat flux)/ $\sigma(T_{lw}^4 - T_{rw}^4)$;
T_{lw}, T_{rw},	left wall, right wall temperatures;
x,	distance measured from the left wall;
z,	τ/τ_L.

Greek symbols

α,	volumetric absorption coefficient;
$\Delta_1(\tau_L)$,	coefficient in the inner expansion;
Θ,	dimensionless emissive power, $[T^4(x) - T_{rw}^4]/[T_{lw}^4 - T_{rw}^4]$;
$\Theta^{(i)}$, $\Theta^{(O)}$, $\Theta^{(c)}$,	inner, outer, and composite expansion for Θ;
σ,	Stefan–Boltzmann constant;
τ,	optical distance;
τ_L,	optical thickness of the layer

INTRODUCTION

It is well known that when a gas is optically thick, the radiant heat flux is accurately given by the Rosseland diffusion approximation. This approximation, however, is invalid close to a wall. A temperature-jump wall-boundary condition is usually used to improve the approximation. Deissler [1] (also see [2]) obtains a second-order jump condition by means of a Taylor series expansion for the emissive power. As Heaslet and Warming [3, 4] noted, however, the emissive power (or temperature) is non-analytic close to the wall. Consequently, a jump condition based on such an expansion is open to question.

The purpose of this note is to examine by the method of matched asymptotic expansions (MAE) the radiant transfer in an optically thick absorbing and emitting grey gas situated between parallel, black, diffusely-emitting walls. It is not necessary to consider more general wall conditions, since the problem for such conditions has been shown [5, 6] to reduce to the case studied here. We further assume that the gas is motionless, that local thermodynamic equilibrium prevails, and that heat conduction can be ignored. These assumptions, which are now fairly standard, are introduced in order to simplify the mathematics without losing, for our purposes, any essential physical features.

We shall show that when the gas is optically thick there is a thin radiative layer adjacent to each wall, analogous to a Knudsen layer in kinetic theory. The solution for this radiative layer in nonanalytic, and the aforementioned Taylor series expansion is, in fact, not valid in it. We shall also show a close connection between the solution for this layer and that for the restricted Milne problem of astrophysics for grey gas [7].

Previous applications of MAE apparently have been limited to problems involving both heat conduction and radiation, as in [2, 8, 9]. The presence of heat conduction introduces additional parameters, thereby masking the radiative wall layer. Furthermore, problems involving radiative energy transport, with or without other energy transport mechanism, frequently make use of the exponential kernel approximation (or the differential or Milne–Eddington approximations). We will show that once such an approximation is made the radiative layer disappears.

FORMULATION

We consider two walls separated by a distance L. The dimensionless optical distance τ is

$$\tau(x) = \int_0^x \alpha(x_1)\,dx_1, \tag{1}$$

where the distance x is measured perpendicular to the left wall and α is the volumetric absorption coefficient. The optical thickness of the layer is $\tau_L = \tau(L)$. Our object is to find a uniformly valid solution, when τ_L is large, for the non-dimensional emissive power Θ and heat flux Q. The basic equations are [3]

$$Q = 2E_3(\tau) + 2\int_0^\tau \Theta(\tau_1)\,E_2(\tau - \tau_1)\,d\tau_1 - 2\int_\tau^{\tau_L} \Theta(\tau_1) \times E_2(\tau_1 - \tau)\,d\tau_1 \tag{2}$$

* This work was performed under Air Force Contract F04695-67-C-0158.

$$\Theta(\tau) = \tfrac{1}{2}E_2(\tau) + \tfrac{1}{2}\int_0^{\tau_L} \Theta(\tau_1)\, E_1(|\tau - \tau_1|)\, d\tau_1, \tag{3}$$

where Q is a constant and equation (3) is obtained by differentiating (2) with respect to τ. (See Nomenclature for all definitions.) The problem can also be formulated in terms of the differential transport equation for the intensity [9]. The foregoing integral equation formulation, however, is advantageous since it involves only the quantities of primary engineering interest. Furthermore, it is easy to apply the method of MAE to equations (2) and (3),* and the resulting solution is quite simple.

Both Θ and Q depend on τ_L, the only parameter in the dimensionless problem. We also make use of the properties, first shown by Meghreblian [10], that $\Theta(\tau_L/2) = \frac{1}{2}$ and that $\Theta - \frac{1}{2}$ is antisymmetric about $\tau = \tau_L/2$. Note that if α is assumed constant, the subsequent analysis is somewhat simplified [11]

OUTER EXPANSION

We refer to the region between the walls, exclusive of the thin thermal wall layers, as the outer region. In this region, Θ and $z(= \tau/\tau_L)$ are of order unity. We here refer to an expansion as complete if it differs from its exact value by at most exponentially small terms. Let $\Theta^{(O)}(z;\,\tau_L)$ be the complete outer asymptotic expansion of Θ for large τ_L, i.e. Θ and $\Theta^{(O)}$ differ in the outer region by at most exponentially small terms. We assume $\Theta^{(O)}$ to be analytic for $0 \leqslant z \leqslant 1$, even though it is not necessarily a valid representation for Θ in the vicinity of the walls. This situation is directly analogous to boundary-layer theory [12]. The most direct way to obtain an equation for $\Theta^{(O)}$ is to replace Θ in (3) by $\Theta^{(O)}$ and then to integrate this equation repeatedly by parts. The difference $\Theta - \Theta^{(O)}$ is of order τ_L^{-1} near the walls, as can be deduced from results shown below. The kernel in (3), however, is exponentially small in these regions, and therefore the use of $\Theta^{(O)}$ in (3) introduces only an exponentially small error. After two integrations, for example, we obtain

$$\tau_L[\Theta^{(O)}(0) - 1]E_2(\tau_L z) + \tau_L\Theta^{(O)}(1)\, E_2[\tau_L(1 - z)]$$

$$- \frac{d\Theta^{(O)}(0)}{dz} E_3(\tau_L z) + \frac{d\Theta^{(O)}(1)}{dz} E_3[\tau_L(1 - z)]$$

$$= \int_0^1 \frac{d^2\Theta^{(O)}}{dz_1^2} E_3(\tau_L|z - z_1|)\, dz_1, \tag{4}$$

where τ_L has been replaced by the outer variable z and $\Theta^{(O)}(1) = \Theta^{(O)}\,(z = 1;\,\tau_L)$, etc. When τ_L is large, the terms on the left-hand side are exponentially small outside of the wall layers (see [13] for properties of the E_n functions). The E_3 function in the integrand is also exponentially small, except in the vicinity of $z = z_1$, where it peaks sharply. Hence, the integral may be approximated by

$$\frac{d^2\Theta^{(O)}(z)}{dz^2}\int_0^1 E_3(\tau_L|z - z_1|)\, dz_1$$

$$= \frac{1}{\tau_L}(\tfrac{2}{3} - \{E_4(\tau_L z) + E_4[\tau_L(1 - z)]\})\frac{d^2\Theta^{(O)}}{dz^2} \tag{5}$$

and the terms within the braces in (5) are exponentially small. [Some care must be exercised when applying the foregoing approximation. For example, it cannot be directly applied to equation (2) due to the difference in sign of the two integrals.] We conclude, therefore, that the equation for the complete outer expansion is given by

$$\frac{d^2\Theta^{(O)}}{dz^2} = 0. \tag{6}$$

Further integration by parts of (4), coupled with the foregoing reasoning, confirms (6). The solution of (6) is

$$\Theta^{(O)}(z) = \tfrac{1}{2} - \tfrac{1}{2}A + Az, \tag{7}$$

where $\Theta^{(O)}(\frac{1}{2}) = \frac{1}{2}$ determines one integration constant, and A depends on τ_L in some still undetermined manner.

Normally, $\Theta^{(O)}$ would be written as $\Theta_0^{(O)}(z) + \delta_1(\tau_L) \times \Theta_1^{(O)}(z) + \ldots$, but this procedure is not necessary here. Since (7) is the complete outer expansion, it equals the "inner expansion of the outer one" ([12], p. 92), designated by $[\Theta^{(O)}]^{(i)}$. With $\Theta^{(O)} = [\Theta^{(O)}]^{(i)}$, the inner expansion is then a composite, uniformly valid in both the inner and outer regions. This fact simplifies our treatment of the inner expansion, since the use of a composite inside the integrals is less questionable than the use of a possibly nonuniformly valid inner expansion.

INNER EXPANSION

Without loss of generality, we consider only the layer adjacent to the left wall. In this inner region, Θ and the optical thickness τ are of order unity, and Q is known to be small. With this in mind, we see that equation (2) consists of terms of order unity, such as $2E_3(\tau)$, and terms of the same order as Q. A two term inner expansion*

$$\Theta^{(i)}(\tau;\,\tau_L) = \Theta_0^{(i)}(\tau) + \Delta_1(\tau_L)\Theta_1^{(i)}(\tau) \tag{8}$$

is thus adequate, where Δ_1 is of the same order as Q. We introduce (8) into (2) and then let $\tau_L \to \infty$ (this produces at

* To the author's knowledge, this note represents the first application of the method of MAE to the integral equations of radiative transfer.

* For purposes of clarity, we refer to $\Theta^{(i)}$ as the inner expansion rather than a composite one, since we still have to match the inner and outer expansions.

most an exponentially small error) to obtain

$$E_3(\tau) = -\int_0^\tau \Theta_0^{(i)}(\tau_1)\,E_2(\tau-\tau_1)\,d\tau_1 + \int_\tau^L \Theta_0^{(i)}(\tau_1) \times E_2(\tau_1-\tau)\,d\tau_1, \tag{9a}$$

$$1 = -\int_0^\tau \Theta_1^{(i)}(\tau_1)\,E_2(\tau-\tau_1)\,d\tau_1 + \int_\tau^\infty \Theta_1^{(i)}(\tau_1) \times E_2(\tau_1-\tau)\,d\tau_1, \tag{9b}$$

provided we set $\Delta_1(\tau_L) = -Q/2$. The solution for (9a) is readily shown to be $\Theta_0^{(i)}(\tau) = 1$. Equation (9b) is Milne's second integral equation ([7], p. 35 with $F = 2$). The exact solution is ([7], p. 138)

$$\Theta_1^{(i)}(\tau) = \tfrac{3}{2}[\tau + q(\tau)],$$

where $q(\tau)$ is Hopf's function, which is tabulated to six significant figures in Table XXXIII of [7]. For later use we note that $q(0) = 3^{-\frac{1}{2}} = 0{\cdot}577351$ and $q(\infty) = 0{\cdot}710447$. By combining the foregoing, we have the inner expansion

$$\Theta^{(i)} = 1 - \tfrac{3}{4}Q[\tau + q(\tau)], \tag{10}$$

where the nonanalytic behaviour of $\Theta^{(i)}$ is contained in $q(\tau)$. For large τ, $q(\tau) = q(\infty) +$ exponentially small terms ([7], p. 198), and hence $\Theta^{(i)}$ is the complete left-wall inner expansion.

RESULTS AND DISCUSSION

Instead of the usual procedure of matching the inner and outer expansions term by term when they are written as series in inverse powers of τ_L, it is a simple matter to match equations (7) and (10) directly. To do this, we replace z by τ/τ_L in (7) and set $q(\tau) = q(\infty)$ in (10) and compare the resulting equations. We thereby obtain

$$A = -\frac{\tau_L}{2q(\infty) + \tau_L}, \tag{11a}$$

$$Q = \frac{1}{\tfrac{3}{2}q(\infty) + \tfrac{3}{4}\tau_L}. \tag{11b}$$

Our results for a composite Θ, valid when $0 \leqslant \tau \leqslant \tau_L/2$, and the emissive power slip at the left wall are summarized as follows:

$$\Theta^{(c)} = 1 - \frac{\tau + q(\tau)}{2q(\infty) + \tau_L}, \tag{12a}$$

$$1 - \Theta^{(c)}(0) = \frac{q(0)}{2q(\infty) + \tau_L}. \tag{12b}$$

By means of the antisymmetry property of Θ, a composite Θ valid for all τ is readily shown to be

$$\Theta^{(c)} = \frac{q(\infty) - q(\tau) + q(\tau_L - \tau) + \tau_L - \tau}{2q(\infty) + \tau_L}. \tag{13}$$

It should be noted that Q and $\Theta^{(c)}$ are not only uniformly valid but are also complete in the sense defined earlier. These quantities can be further improved only by the inclusion of exponentially small terms.

Equations (11b) and (12b) are identical to equations (53a) and (56), respectively, in [3], which are based on expansions of the X and Y functions of Chandrasekhar. This agreement is a further demonstration of the excellent results obtainable by the method of MAE, which is frequently simpler than other methods. Equation (13) is identical to equation (A20) in [4], which is obtained by Case's method [14] of normal mode expansion. Heaslet and Warming [3] give exact numerical values for Θ, with $\tau_L = 10$, in their Fig. 2. Values of $\Theta^{(c)}$ calculated by (13) fall exactly on this curve.

A comparison of Q and $\Theta^{(c)}$ with equation (26) and Table 2 in [4] show that our asymptotic results are rather accurate even for small τ_L. For example, equation (11b) gives 0·94 for Q when $v_L = 0$ as compared with an exact value of unity. This agreement occurs only for the plane parallel layer and is thus fortuitous. A recently completed study by the author of the radiative heat transfer between concentric spheres demonstrates that our asymptotic method is generally valid only when $\tau_L \gg 1$, as would be expected.

The Milne Eddington approximation ([7], p. 88) corresponds to $q(\tau) = \tfrac{2}{3}$. With this values for $q(\tau)$, equations (11b) and (12b) become identical to Deissler's [1] results. This approximation is excellent for Q since $\tfrac{3}{2}q(\infty)$ differs only slightly from unity. The emissive power slip, however, is poorly given as previously noted in [3]. The approximation, $q(\tau) =$ constant, not only neglects the nonanalytic behavior of Θ near the wall but also removes the distinction between the inner and outer regions, since both expansions are then linear in τ. The fact that approximations, e.g. Milne–Eddington, on the whole give adequate results is due to the relatively small variation in $q(\tau)$, which increase monotonically from $q(0)$ to $q(\infty)$.

ACKNOWLEDGEMENT

The author acknowledges with pleasure the helpful discussions with Dr. Melvin Epstein of Aerospace Corporation and the useful comments of one of the referees.

REFERENCES

1. R. G. Deissler, Diffusion approximation for thermal radiation in gases with jump boundary condition, *J. Heat Transfer* **86**, 240–246 (1964).
2. E. M. Sparrow and R. D. Cess, *Radiation Heat Transfer*, Chapters 8 and 9, Brooks/Cole, Belmont, Calif. (1966).
3. M. A. Heaslet and R. F. Warming, Radiative transport and wall temperature slip in an absorbing planar medium, *Int. J. Heat Mass Transfer* **8**, 979–994 (1965).
4. M. A. Heaslet and R. F. Warming, Radiative transport

in an absorbing planar medium II. Predictions of radiative source functions, *Int. J. Heat Mass Transfer* **10**, 1413–1427 (1967).
5. R. VISKANTA and R. J. GROSH, Heat transfer in a thermal radiation absorbing and scattering medium, *Proceedings of the International Heat Transfer Conference*, p. 820. Am. Soc. Mech. Engrs, New York (1961).
6. M. A. HEASLET and F. B. FULLER, Approximate predictions of the transport of thermal radiation through an absorbing plane layer, NASA TN D-2515 (1964).
7. V. KOURGANOFF, *Basic Methods in Transfer Problems*. Dover, New York (1963).
8. W. LICK, Energy transfer by radiation and conduction, *Proc. Heat Transf. Fluid Mech. Inst.*, 14–26 (1963).
9. J. WALLACE, The opaque limit in radiation gas dynamics, AIAA Paper No. 65–80 (1965).
10. R. V. MEGHREBLIAN, An approximate analytical solution for the radiation exchange between two flat surfaces separated by an absorbing gas, *Int. J. Mass Transfer* **5**, 1051–1052 (1962).
11. G. EMANUEL, Radiative transport in an optically thick planar medium, Aerospace Corp. Report TR-0158 (3240-20)-7 (1967).
12. M. VAN DYKE, *Perturbation Methods in Fluid Mechanics*, Chapter VII. Academic Press, New York (1964).
13. M. ABRAMOWITZ and I. A. STEGUN, editors, *Handbook of Mathematical Functions*, Chapter 5. National Bureau of Standards, Washington, D.C. (1964).
14. J. H. FERZIGER and G. M. SIMMONS, Application of Case's method to plane parallel radiative transfer, *Int. J. Heat Mass Transfer* **9**, 987–992 (1966).